

GLOBAL WELL-POSEDNESS FOR HIGHER-ORDER SCHRÖDINGER EQUATIONS IN WEIGHTED L^2 SPACES

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ABSTRACT. We obtain the global well-posedness for Schrödinger equations of higher orders in weighted L^2 spaces. This is based on weighted L^2 Strichartz estimates for the corresponding propagator with higher-order dispersion. Our method is also applied to the Airy equation which is the linear component of Korteweg-de Vries type equations.

1. INTRODUCTION

In this paper we are concerned with global well-posedness of the following Cauchy problem for perturbed Schrödinger equations of higher orders $a > 2$:

$$\begin{cases} i\partial_t u + (-\Delta)^{a/2} u + V(x, t)u = F(x, t), \\ u(x, 0) = u_0(x), \end{cases} \quad (1.1)$$

where $(x, t) \in \mathbb{R}^{n+1}$, $n \geq 1$, and $(-\Delta)^{a/2}$ is given by means of the Fourier transform $\mathcal{F}f (= \widehat{f})$:

$$\mathcal{F}[(-\Delta)^{a/2} f](\xi) = |\xi|^a \widehat{f}(\xi).$$

Nowadays, the time-dependent Schrödinger operator $i\partial_t u - \Delta$ became one of the most popular differential operators of modern mathematics as well as physics. The higher-order counterpart of it has been also attracted for decades from mathematical physics. For instance, the forth-order ($a = 4$) dispersion term Δ^2 has been used in the formation and propagation of intense laser beams in a bulk medium with a nonlinearity f , as follows ([12, 13]):

$$i\partial_t u + \delta \Delta^2 u = f(|u|)u,$$

where $\delta > 0$ or $\delta < 0$. Global well-posedness of solutions for this PDE model has been studied in Sobolev spaces (see, for example, [22, 23, 24, 33]).

Our work here is aimed at finding a suitable condition on the perturbed term $V(x, t)$ which guarantees existence and uniqueness of solutions to (1.1) in the weighted L^2 space, $L^2(|V|dxdt)$, if $u_0 \in L^2$ and $F \in L^2(|V|^{-1}dxdt)$. Furthermore, it turns out that the solution u belongs to $C_t L_x^2$.

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We will consider a function class, denoted by $\mathfrak{L}^{\alpha,\beta,p}$, of V to suit our purpose, which is defined by

$$\|V\|_{\mathfrak{L}^{\alpha,\beta,p}} := \sup_{(x,t) \in \mathbb{R}^{n+1}, r, l > 0} r^\alpha l^\beta \left(\frac{1}{r^n l} \int_{Q(x,r) \times I(t,l)} |V(y,s)|^p dy ds \right)^{\frac{1}{p}} < \infty$$

for $0 < \alpha \leq n/p$ and $0 < \beta \leq 1/p$. Here, $Q(x,r)$ denotes a cube in \mathbb{R}^n centered at x with side length r , and $I(t,l)$ denotes an interval in \mathbb{R} centered at t with length l . From the definition, it is an elementary matter to check that $\mathfrak{L}^{\alpha,\beta,p}$ has the dilation property $\|V(\lambda \cdot, \lambda^a \cdot)\|_{\mathfrak{L}^{\alpha,\beta,p}} = \lambda^{-\alpha} \lambda^{-a\beta} \|V\|_{\mathfrak{L}^{\alpha,\beta,p}}$. It should be noted that when $r = l$, the above class is just the same as the usual Morrey-Campanato class $\mathfrak{L}^{\alpha+\beta,p}(\mathbb{R}^{n+1})$. Also, when $r = \sqrt{l}$, it becomes equivalent to the so-called parabolic Morrey-Campanato class $\mathfrak{L}_{par}^{\alpha+2\beta,p}(\mathbb{R}^{n+1})$ introduced in [1]. Furthermore, $\mathfrak{L}^{\beta,p}(\mathbb{R}; \mathfrak{L}^{\alpha,p}(\mathbb{R}^n)) \subset \mathfrak{L}^{\alpha,\beta,p}(\mathbb{R}^{n+1})$. Recall that $\mathfrak{L}^{\alpha,p} = L^p$ when $p = n/\alpha$, and even $L^{n/\alpha,\infty} \subset \mathfrak{L}^{\alpha,p}$ for $p < n/\alpha$.

Our well-posedness result for the above Cauchy problem is stated as follows:

Theorem 1.1. *Let $n \geq 1$ and $a > (n+2)/2$. Assume that $V \in \mathfrak{L}^{\alpha,\beta,p}$ with $\|V\|_{\mathfrak{L}^{\alpha,\beta,p}}$ small enough for $1 < p < 2$, $a = \alpha + a\beta$ and $\alpha + \beta \geq (n+2)/2$. Then, if $u_0 \in L^2$ and $F \in L^2(|V|^{-1})$, there exists a unique solution of the problem (1.1) in the space $L^2(|V|)$. Furthermore, the solution u belongs to $C_t L_x^2$ and satisfies the following inequalities:*

$$\|u\|_{L^2(|V|)} \leq C \|V\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2} \|u_0\|_{L^2} + C \|V\|_{\mathfrak{L}^{\alpha,\beta,p}} \|F\|_{L_{t,x}^2(|V|^{-1})} \quad (1.2)$$

and

$$\sup_{t \in \mathbb{R}} \|u\|_{L_x^2} \leq C \|u_0\|_{L^2} + C \|V\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2} \|F\|_{L_{t,x}^2(|V|^{-1})}. \quad (1.3)$$

A natural way to achieve this result is to obtain weighted L^2 estimates for the solutions in terms of the initial datum u_0 and the forcing term F . To be precise, let us first consider the following non-perturbed problem:

$$\begin{cases} i\partial_t u + (-\Delta)^{a/2} u = F(x, t), \\ u(x, 0) = f(x). \end{cases}$$

As is well known, the solution is then given by

$$u(x, t) = e^{it(-\Delta)^{a/2}} f(x) - i \int_0^t e^{i(t-s)(-\Delta)^{a/2}} F(\cdot, s) ds,$$

where the evolution operator $e^{it(-\Delta)^{a/2}}$ is defined by

$$e^{it(-\Delta)^{a/2}} f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|^a} \widehat{f}(\xi) d\xi.$$

The proof of Theorem 1.1 will be done in the next section by making use of the following weighted L^2 Strichartz estimates:

Theorem 1.2. *Let $n \geq 1$ and $a > (n+2)/2$. Assume that $w \geq 0$ is a function in $\mathfrak{L}^{\alpha,\beta,p}$. Then we have*

$$\|e^{it(-\Delta)^{a/2}} f\|_{L^2(w(x,t))} \leq C \|w\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2} \|f\|_{L^2}, \quad (1.4)$$

and

$$\left\| \int_0^t e^{i(t-s)(-\Delta)^{a/2}} F(\cdot, s) ds \right\|_{L^2(w(x,t))} \leq C \|w\|_{\mathfrak{L}^{\alpha, \beta, p}} \|F\|_{L^2(w(x,t)^{-1})} \quad (1.5)$$

if

$$1 < p < 2, \quad a = \alpha + a\beta \quad \text{and} \quad \alpha + \beta \geq (n+2)/2. \quad (1.6)$$

Remark 1.3. The condition $a = \alpha + a\beta$ in the theorem is needed for the scaling invariance of the estimates (1.4) and (1.5) under the scaling $(x, t) \rightarrow (\lambda x, \lambda^a t)$, $\lambda > 0$.

Remark 1.4. Since $0 < \alpha \leq n/p$ and $0 < \beta \leq 1/p$, it is not difficult to check that for $n \geq 1$ and $a > (n+2)/2$, there exist α , β and p satisfying the condition (1.6) as follows:

$$\begin{aligned} 1 < p &\leq \min\left\{\frac{2(a-1)}{a}, \frac{a+n}{a}\right\}, \\ \max\left\{\frac{an}{2(a-1)}, \frac{a}{p}\right\} &\leq \alpha \leq \frac{n}{p}, \\ 1 - \frac{n}{ap} &\leq \beta \leq \min\left\{\frac{n+2-2a}{2(1-a)}, \frac{1}{p}\right\}. \end{aligned}$$

At this point, we would like to point out that when $n = 1$, our theorems can also cover the fractional order cases where $3/2 < a < 2$.

The Strichartz estimates for the Schrödinger equation ($a = 2$) in the usual $L_t^q L_x^r$ norms have been extensively developed in the works [30, 10, 4, 14, 15, 9, 32, 16, 19]. See also [7, 8, 26, 20] for related results. In the weighted L^2 setting as above, they were also studied in [25, 2, 28] using weighted L^2 resolvent estimates for the Laplacian, and were applied to the problem of well-posedness for the Schrödinger equation in weighted L^2 spaces. Our method here for higher orders $a > 2$ is entirely different from them and is inspired by [1] and our earlier work [17] for the wave operator $-\partial_t^2 + \Delta$. It will be based on a combination of a localization argument in weighted L^2 spaces and a bilinear interpolation argument. See the final section, Section 4. The key ingredient in doing so is Lemma 3.2 in Section 3 which enables us to make use of frequency localized estimates (see Proposition 4.1) in weighted L^2 spaces without additional assumption on the weight w .

Let us now mention an implication of Theorem 1.2 for the Airy equation which is the linear component of Korteweg-de Vries type equations. First, consider the corresponding problem

$$\begin{cases} \partial_t u + \partial_x^3 u = F(x, t), \\ u(x, 0) = f(x), \end{cases} \quad (1.7)$$

where $(x, t) \in \mathbb{R}^{1+1}$. Using the Fourier transform, the solution is then written as

$$u(x, t) = e^{-t\partial_x^3} f(x) + \int_0^t e^{-(t-s)\partial_x^3} F(\cdot, s) ds.$$

Here we note that the operator $e^{-t\partial_x^3}$ is given by

$$e^{-t\partial_x^3} f(x) = \int_{\mathbb{R}^1} e^{ix\xi} e^{it\xi^3} \widehat{f}(\xi) d\xi.$$

Thus, the proof of Theorem 1.2 when $n = 1$ and $a = 3$ would be clearly worked for (1.7). Then, from this observation and Remark 1.4, the resulting estimates would be given as follows:

Theorem 1.5. *Assume that $w \geq 0$ is a function in $\mathfrak{L}^{\alpha,\beta,p}(\mathbb{R})$. Then we have*

$$\|e^{-t\partial_x^3} f\|_{L^2(w(x,t))} \leq C \|w\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2} \|f\|_{L^2},$$

and

$$\left\| \int_0^t e^{-(t-s)\partial_x^3} F(\cdot, s) ds \right\|_{L^2(w(x,t))} \leq C \|w\|_{\mathfrak{L}^{\alpha,\beta,p}} \|F\|_{L^2(w(x,t)^{-1})}$$

for $1 < p \leq 4/3$, $3/4 \leq \alpha \leq 1/p$ and $1 - 1/(3p) \leq \beta \leq 3/4$.

Also, from the same argument for Theorem 1.1, this theorem implies the following global well-posedness for the corresponding Cauchy problem

$$\begin{cases} \partial_t u + \partial_x^3 u + V(x, t)u = F(x, t), \\ u(x, 0) = u_0(x). \end{cases} \quad (1.8)$$

Corollary 1.6. *Assume that $V \in \mathfrak{L}^{\alpha,\beta,p}$ with $\|V\|_{\mathfrak{L}^{\alpha,\beta,p}}$ small enough for $1 < p \leq 4/3$, $3/4 \leq \alpha \leq 1/p$ and $1 - 1/(3p) \leq \beta \leq 3/4$. Then, if $u_0 \in L^2$ and $F \in L^2(|V|^{-1})$, there exists a unique solution of the problem (1.8) in the space $L^2(|V|)$. Furthermore, the solution u belongs to $C_t L_x^2$ and satisfies the inequalities (1.2) and (1.3).*

Throughout this paper, the letter C stands for a constant which may be different at each occurrence. Also, we denote by \widehat{f} the Fourier transform of f and by $\langle f, g \rangle$ the usual inner product of f, g on L^2 .

2. PROOF OF THEOREM 1.1

In this section we explain how to deduce the well-posedness (Theorem 1.1) for the Cauchy problem (1.1) from the weighted L^2 Strichartz estimates in Theorem 1.2.

The starting point is that the solution of (1.1) is given by the following integral equation

$$u(x, t) = e^{it(-\Delta)^{a/2}} u_0(x) - i \int_0^t e^{i(t-s)(-\Delta)^{a/2}} F(\cdot, s) ds + \Phi(u)(x, t), \quad (2.1)$$

where

$$\Phi(u)(x, t) = -i \int_0^t e^{i(t-s)(-\Delta)^{a/2}} (Vu)(\cdot, s) ds.$$

Note here that

$$(I - \Phi)(u) = e^{it(-\Delta)^{a/2}} u_0(x) - i \int_0^t e^{i(t-s)(-\Delta)^{a/2}} F(\cdot, s) ds,$$

where I is the identity operator. Then, since $u_0 \in L^2$ and $F \in L^2(|V|^{-1})$, from the weighted L^2 Strichartz estimates in Theorem 1.2 with $w = |V|$, it follows that

$$(I - \Phi)(u) \in L^2(|V|).$$

Hence, for the global existence of the solution, we want to show that the operator $I - \Phi$ has an inverse in the space $L^2(|V|)$. This is valid if the operator norm for Φ in $L^2(|V|)$ is strictly less than 1. Namely, we need to show $\|\Phi(u)\|_{L^2(|V|)} < \|u\|_{L^2(|V|)}$. But, from the inhomogeneous estimate (1.5) with $w = |V|$, we see that

$$\begin{aligned} \|\Phi(u)\|_{L^2(|V|)} &\leq C\|V\|_{\mathfrak{L}^{\alpha,\beta,p}}\|Vu\|_{L^2(|V|^{-1})} \\ &= C\|V\|_{\mathfrak{L}^{\alpha,\beta,p}}\|u\|_{L^2(|V|)} \\ &< \frac{1}{2}\|u\|_{L^2(|V|)} \end{aligned} \quad (2.2)$$

by the smallness assumption on the norm $\|V\|_{\mathfrak{L}^{\alpha,\beta,p}}$.

On the other hand, from (2.1), (2.2) and Theorem 1.2, it follows easily that

$$\begin{aligned} \|u\|_{L^2(|V|)} &\leq C\|e^{it(-\Delta)^{a/2}}u_0\|_{L^2(|V|)} + C\left\|\int_0^t e^{i(t-s)(-\Delta)^{a/2}}F(\cdot, s)ds\right\|_{L^2(|V|)} \\ &\leq C\|V\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2}\|u_0\|_{L^2} + C\|V\|_{\mathfrak{L}^{\alpha,\beta,p}}\|F\|_{L^2_{t,x}(|V|^{-1})}. \end{aligned} \quad (2.3)$$

Hence (1.2) is proved. To show (1.3), we will use (2.3) and the following estimate

$$\left\|\int_{-\infty}^{\infty} e^{-is(-\Delta)^{a/2}}F(\cdot, s)ds\right\|_{L^2_x} \leq C\|w\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2}\|F\|_{L^2(w(x,t)^{-1})} \quad (2.4)$$

which is the dual version of (1.4). First, from (2.1), (2.4) with $w = |V|$, and the simple fact that $e^{it(-\Delta)^{a/2}}$ is an isometry in L^2 , one can see that

$$\|u\|_{L^2_x} \leq C\|u_0\|_{L^2} + C\|V\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2}\|F\|_{L^2(|V|^{-1})} + C\|V\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2}\|Vu\|_{L^2(|V|^{-1})}.$$

Since $\|Vu\|_{L^2(|V|^{-1})} = \|u\|_{L^2(|V|)}$ and $\|V\|_{\mathfrak{L}^{\alpha,\beta,p}}$ is small, from this and (2.3), it follows now that

$$\|u\|_{L^2_x} \leq C\|u_0\|_{L^2} + C\|V\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2}\|F\|_{L^2_{t,x}(|V|^{-1})}.$$

So we get (1.3). This completes the proof.

3. PRELIMINARIES

Here we present some preliminary lemmas which are needed in the next section for the proof of Theorem 1.2.

Given two complex Banach spaces A_0 and A_1 , for $0 < \theta < 1$ and $1 \leq q \leq \infty$, we denote by $(A_0, A_1)_{\theta,q}$ the real interpolation spaces equipped with the norms

$$\|a\|_{(A_0, A_1)_{\theta, \infty}} = \sup_{0 < t < \infty} t^{-\theta} K(t, a)$$

and

$$\|a\|_{(A_0, A_1)_{\theta, q}} = \left(\int_0^\infty (t^{-\theta} K(t, a))^q dt \right)^{1/q}, \quad 1 \leq q < \infty,$$

where

$$K(t, a) = \inf_{a=a_0+a_1} \|a_0\|_{A_0} + t\|a_1\|_{A_1}$$

for $0 < t < \infty$ and $a \in A_0 + A_1$. In particular, $(A_0, A_1)_{\theta, q} = A_0 = A_1$ if $A_0 = A_1$. See [3, 31] for details. The following bilinear interpolation lemma concerning these interpolation spaces is well-known (see [3], Section 3.13, Exercise 5(b)).

Lemma 3.1. *For $i = 0, 1$, let A_i, B_i, C_i be Banach spaces and let T be a bilinear operator such that*

$$T : A_0 \times B_0 \rightarrow C_0,$$

$$T : A_0 \times B_1 \rightarrow C_1,$$

$$T : A_1 \times B_0 \rightarrow C_1.$$

Then one has for $\theta = \theta_0 + \theta_1$ and $1/q + 1/r \geq 1$

$$T : (A_0, A_1)_{\theta_0, q} \times (B_0, B_1)_{\theta_1, r} \rightarrow (C_0, C_1)_{\theta, 1}.$$

Here $0 < \theta_i < \theta < 1$ and $1 \leq q, r \leq \infty$.

Let us now recall that a weight¹ $w : \mathbb{R}^n \rightarrow [0, \infty]$ is said to be in the Muckenhoupt $A_2(\mathbb{R}^n)$ class if there is a constant C_{A_2} such that

$$\sup_{Q \text{ cubes in } \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1} dx \right) < C_{A_2}.$$

(See, for example, [11].) In the following we obtain a useful property of a weight in the space $\mathfrak{L}^{\alpha, \beta, p}$. For some similar properties of the usual Morrey-Campanato spaces, we refer the reader to [5, 17]. Such property has been used earlier in [5, 27, 28] concerning unique continuation for Schrödinger equations.

Lemma 3.2. *For a weight $w \in \mathfrak{L}^{\alpha, \beta, p}$ on \mathbb{R}^{n+1} , let $w_*(x, t)$ be the n -dimensional maximal function given by*

$$w_*(x, t) = \sup_{Q'} \left(\frac{1}{|Q'|} \int_{Q'} w(y, t)^p dy \right)^{\frac{1}{p}}, \quad \rho > 1,$$

where Q' denotes a cube in \mathbb{R}^n with center x . Then, if $p > \rho$ we have $\|w_\|_{\mathfrak{L}^{\alpha, \beta, p}} \leq C\|w\|_{\mathfrak{L}^{\alpha, \beta, p}}$, and $w_*(\cdot, t) \in A_2(\mathbb{R}^n)$ in the x variable with a constant C_{A_2} uniform in almost every $t \in \mathbb{R}$.*

Proof. To show $\|w_*\|_{\mathfrak{L}^{\alpha, \beta, p}} \leq C\|w\|_{\mathfrak{L}^{\alpha, \beta, p}}$, we first fix a cube $Q(z, r) \times I(\tau, l)$ in \mathbb{R}^{n+1} . Then, we define the rectangles R_k , $k \geq 1$, such that $(y, t) \in R_k$ if $|t - \tau| < 2l$ and $y \in Q(z, 2^{k+1}r) \setminus Q(z, 2^k r)$, and set $R_0 = Q(z, 4r) \times I(\tau, 4l)$.

Now we can write

$$w(y, t) = \sum_{k \geq 0} w^{(k)}(y, t) + \phi(y, t),$$

where $w^{(k)} = w\chi_{R_k}$ with the characteristic function χ_{R_k} of the set R_k , and $\phi(y, t)$ is a function supported on $\mathbb{R}^{n+1} \setminus \bigcup_{k \geq 0} R_k$. Also it is not difficult to see that

$$w_*(x, t) \leq \sum_{k \geq 0} w_*^{(k)}(x, t) + \phi_*(x, t)$$

¹ It is a locally integrable function which is allowed to be zero or infinite only on a set of Lebesgue measure zero.

and

$$\begin{aligned} \left(\int_{Q(z,r) \times I(\tau,l)} w_*(x,t)^p dx dt \right)^{\frac{1}{p}} &\leq \sum_{k \geq 0} \left(\int_{Q(z,r) \times I(\tau,l)} w_*^{(k)}(x,t)^p dx dt \right)^{\frac{1}{p}} \\ &\quad + \left(\int_{Q(z,r) \times I(\tau,l)} \phi_*(x,t)^p dx dt \right)^{\frac{1}{p}}. \end{aligned}$$

Since $(x,t) \in Q(z,r) \times Q(\tau,l)$, it is clear that $\phi_*(x,t) = 0$. For the term where $k = 0$, we use the following well-known maximal theorem

$$\|M(f)\|_q \leq C\|f\|_q, \quad q > 1,$$

where $M(f)$ is the usual Hardy-Littlewood maximal function defined by

$$M(f)(x) = \sup_Q \frac{1}{|Q|} \int_Q f(y) dy. \quad (3.1)$$

(Here, the sup is taken over all cubes Q in \mathbb{R}^n with center x .) Indeed, by applying the maximal theorem with $q = p/\rho$ in x -variable, one can see that if $p > \rho$

$$\begin{aligned} r^\alpha l^\beta \left(\frac{1}{r^n l} \int_{Q(z,r) \times I(\tau,l)} w_*^{(0)}(x,t)^p dx dt \right)^{\frac{1}{p}} \\ \leq C r^\alpha l^\beta \left(\frac{1}{r^n l} \int_{Q(z,4r) \times I(\tau,4l)} w(y,t)^p dy dt \right)^{\frac{1}{p}} \\ \leq C \|w\|_{\mathfrak{L}^{\alpha,\beta,p}}. \end{aligned} \quad (3.2)$$

So it remains to consider the term where $k \geq 1$.

Let $k \geq 1$. Since $(x,t) \in Q(z,r) \times Q(\tau,l)$, it follows that

$$\begin{aligned} w_*^{(k)}(x,t) &= \sup_{Q' \subset \mathbb{R}^n} \left(\frac{1}{|Q'|} \int_{Q'} w(y,t)^p \chi_{R_k}(y,t) dy \right)^{\frac{1}{p}} \\ &\leq C \left(\frac{1}{(2^k r)^n} \int_{Q(z,2^{k+1}r) \setminus Q(z,2^k r)} w(y,t)^p dy \right)^{\frac{1}{p}} \\ &\leq C \left(\frac{1}{(2^k r)^n} \int_{Q(z,2^{k+1}r) \setminus Q(z,2^k r)} w(y,t)^p dy \right)^{\frac{1}{p}}, \end{aligned}$$

where we used Hölder's inequality for the last inequality since $p \geq \rho$. Hence,

$$\begin{aligned} \int_{Q(z,r) \times I(\tau,l)} w_*^{(k)}(x,t)^p dx dt \\ \leq \frac{C}{(2^k r)^n} \int_{|\tau-t| < l} \int_{Q(z,2^{k+1}r) \setminus Q(z,2^k r)} w(y,t)^p \int_{|z-x| < r} 1 dx dy dt \\ \leq \frac{C}{2^{kn}} \int_{R_k} w(y,t)^p dy dt. \end{aligned}$$

Since $R_k \subset Q(z, 2^{k+1}r) \times I(\tau, 2l)$, this implies that

$$\begin{aligned}
& r^\alpha l^\beta \left(\frac{1}{r^{nl}} \int_{Q(z,r) \times I(\tau,l)} w_*^{(k)}(x,t)^p dx dt \right)^{\frac{1}{p}} \\
& \leq C r^\alpha l^\beta \left(\frac{1}{2^{kn} r^{nl}} \int_{R_k} w(y,t)^p dy dt \right)^{\frac{1}{p}} \\
& \leq C 2^{-\alpha k} (2^k r)^\alpha l^\beta \left(\frac{1}{(2^k r)^{nl}} \int_{Q(z, 2^{k+1}r) \times I(\tau, 2l)} w(y,t)^p dy dt \right)^{\frac{1}{p}} \\
& \leq C 2^{-\alpha k} \|w\|_{\mathfrak{L}^{\alpha, \beta, p}}.
\end{aligned}$$

Hence, since $\alpha > 0$ and $p \geq \rho$, it follows that

$$\sum_{k \geq 1} r^\alpha l^\beta \left(\frac{1}{r^{nl}} \int_{Q(z,r) \times Q(\tau,l)} w_*^{(k)}(x,t)^p dx dt \right)^{\frac{1}{p}} \leq C \|w\|_{\mathfrak{L}^{\alpha, \beta, p}}.$$

Consequently, combining this and (3.2), we get $\|w_*\|_{\mathfrak{L}^{\alpha, \beta, p}} \leq C \|w\|_{\mathfrak{L}^{\alpha, \beta, p}}$ if $p > \rho$, as desired.

Next, for the remaining part ($w_*(\cdot, t) \in A_2(\mathbb{R}^n)$) in the lemma, we first need to recall some known facts for A_1 weights. We say that w is in the class A_1 if there is a constant C_{A_1} such that for almost every x

$$M(w)(x) \leq C_{A_1} w(x),$$

where $M(w)$ is the Hardy-Littlewood maximal function of w (see (3.1)). Then,

$$A_1 \subset A_2 \quad \text{with} \quad C_{A_2} \leq C_{A_1}. \quad (3.3)$$

See, for example, [11] for details. Also, the following fact² is known: If $M(w)(x) < \infty$ for almost every $x \in \mathbb{R}^n$, then for $0 < \delta < 1$

$$(M(w))^\delta \in A_1 \quad (3.4)$$

with C_{A_1} independent of w .

Now we are ready to show that $w_*(\cdot, t) \in A_2(\mathbb{R}^n)$ in the x variable with a constant C_{A_2} uniform in almost every $t \in \mathbb{R}$. Note first that

$$w_*(x, t) = (M(w(\cdot, t)^\rho))^{1/\rho}.$$

Since $w \in \mathfrak{L}^{\alpha, \beta, p}$ and $p \geq \rho$, it is an elementary matter to check that $M(w(\cdot, t)^\rho) < \infty$ for almost every $x \in \mathbb{R}^n$. Then, by applying (3.4) with $\delta = 1/\rho$, it follows that $w_*(\cdot, t) \in A_1$ with C_{A_1} uniform in $t \in \mathbb{R}$. Finally, from (3.3) this immediately implies that $w_*(\cdot, t) \in A_2$ with C_{A_2} uniform in $t \in \mathbb{R}$. \square

²It can be found in Chapter 5 of [29]. See also Proposition 2 in [6].

4. PROOF OF THEOREM 1.2

Since $w \leq w_*$ and $\|w_*\|_{\mathfrak{L}^{\alpha,\beta,p}} \leq C\|w\|_{\mathfrak{L}^{\alpha,\beta,p}}$ for $p > \rho > 1$ (see Lemma 3.2), if we show the homogeneous estimate (1.4) replacing w with w_* , we get

$$\begin{aligned} \|e^{it(-\Delta)^{a/2}} f\|_{L^2(w(x,t))} &\leq \|e^{it(-\Delta)^{a/2}} f\|_{L^2(w_*(x,t))} \\ &\leq C\|w_*\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2} \|f\|_{L^2} \\ &\leq C\|w\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2} \|f\|_{L^2} \end{aligned}$$

as desired. Similarly for the inhomogeneous estimate (1.5). So it suffices to prove Theorem 1.2 by replacing w with w_* . From this replacement we are in a good light that we can use the property $w_*(\cdot, t) \in A_2(\mathbb{R}^n)$ in Lemma 3.2. In fact, this A_2 condition will enable us to make use of a localization argument in weighted L^2 spaces. From the above argument, we shall assume, for simplicity of notation, that w satisfies the same A_2 condition.

Now, let ϕ be a smooth function supported in $(1/2, 2)$ such that

$$\sum_{k=-\infty}^{\infty} \phi(2^k t) = 1, \quad t > 0.$$

Then we define the multiplier operators $P_k f$ for $k \in \mathbb{Z}$ by

$$\widehat{P_k f}(\xi) = \phi(2^{-k}|\xi|)\widehat{f}(\xi).$$

Then we have the following.

Proposition 4.1. *If $\alpha + \beta \geq (n+2)/2$ and $1 < p < 2$, then*

$$\|e^{it(-\Delta)^{a/2}} P_k f\|_{L^2(w(x,t))} \leq C 2^{k(\alpha+a\beta-a)/2} \|w\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2} \|f\|_{L^2} \quad (4.1)$$

and

$$\left\| \int_0^t e^{i(t-s)(-\Delta)^{a/2}} P_k F(\cdot, s) ds \right\|_{L^2(w(x,t))} \leq C 2^{k(\alpha+a\beta-a)/2} \|w\|_{\mathfrak{L}^{\alpha,\beta,p}} \|F\|_{L^2(w(x,t)^{-1})}. \quad (4.2)$$

Assuming this proposition for the moment, we prove Theorem 1.2. First we consider the homogeneous estimate. As mentioned above, since we may assume that $w(\cdot, t) \in A_2(\mathbb{R}^n)$ uniformly for almost every $t \in \mathbb{R}$, by the Littlewood-Paley theorem on weighted L^2 spaces (see Theorem 1 in [18]), we see that

$$\begin{aligned} \|e^{it(-\Delta)^{a/2}} f\|_{L^2(w(x,t))}^2 &= \int \|e^{it(-\Delta)^{a/2}} f\|_{L^2(w(\cdot, t))}^2 dt \\ &\leq C \int \left\| \left(\sum_k |P_k e^{it(-\Delta)^{a/2}} f|^2 \right)^{1/2} \right\|_{L^2(w(\cdot, t))}^2 dt \\ &= C \sum_k \|e^{it(-\Delta)^{a/2}} P_k f\|_{L^2(w(x,t))}^2. \end{aligned}$$

Meanwhile, since $P_k P_j f = 0$ if $|j - k| \geq 2$, it follows from (4.1) that

$$\begin{aligned} \sum_k \|e^{it(-\Delta)^{a/2}} P_k f\|_{L^2(w(x,t))}^2 &= \sum_k \|e^{it(-\Delta)^{a/2}} P_k \left(\sum_{|j-k| \leq 1} P_j f \right)\|_{L^2(w(x,t))}^2 \\ &\leq C \|w\|_{\mathfrak{L}^{\alpha,\beta,p}} \sum_k 2^{k(\alpha+a\beta-a)} \left\| \sum_{|j-k| \leq 1} P_j f \right\|_2^2 \\ &\leq C \|w\|_{\mathfrak{L}^{\alpha,\beta,p}} \|f\|_2^2. \end{aligned}$$

Here, recall that $a = \alpha + a\beta$. Thus, we get

$$\|e^{it(-\Delta)^{a/2}} f\|_{L^2(w(x,t))} \leq C \|w\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2} \|f\|_{L^2}$$

as desired. The inhomogeneous estimate (1.5) follows also from the same argument. Indeed, by the Littlewood-Paley theorem on weighted L^2 spaces as before, one can see that

$$\begin{aligned} &\left\| \int_0^t e^{i(t-s)(-\Delta)^{a/2}} F(\cdot, s) ds \right\|_{L^2(w(x,t))}^2 \\ &\leq C \sum_k \left\| \int_0^t e^{i(t-s)(-\Delta)^{a/2}} P_k \left(\sum_{|j-k| \leq 1} P_j F(\cdot, s) \right) ds \right\|_{L^2(w(x,t))}^2. \end{aligned}$$

By using (4.2), the right-hand side in the above is bounded by

$$C \|w\|_{\mathfrak{L}^{\alpha,\beta,p}}^2 \sum_k 2^{k(\alpha+a\beta-a)} \left\| \sum_{|j-k| \leq 1} P_j F \right\|_{L^2(w(x,t)^{-1})}^2.$$

Since $a = \alpha + a\beta$, and $w(\cdot, t)^{-1} \in A_2(\mathbb{R}^n)$ if and only if $w(\cdot, t) \in A_2(\mathbb{R}^n)$, applying the Littlewood-Paley theorem again, this is bounded by $C \|w\|_{\mathfrak{L}^{\alpha,\beta,p}}^2 \|F\|_{L^2(w(x,t)^{-1})}^2$. Consequently,

$$\left\| \int_0^t e^{i(t-s)(-\Delta)^{a/2}} F(\cdot, s) ds \right\|_{L^2(w(x,t))} \leq C \|w\|_{\mathfrak{L}^{\alpha,\beta,p}} \|F\|_{L^2(w(x,t)^{-1})}.$$

Now we are reduced to showing the proposition, and so the rest of this section will be devoted to it.

Proof of Proposition 4.1. We first show the estimate (4.1). From scaling, it suffices to show the following case where $k = 0$:

$$\|e^{it(-\Delta)^{a/2}} P_0 f\|_{L^2(w(x,t))} \leq C \|w\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2} \|f\|_{L^2}. \quad (4.3)$$

In fact, note that

$$\begin{aligned} \|e^{it(-\Delta)^{a/2}} P_k f\|_{L^2(w(x,t))}^2 &\leq C 2^{-kn} 2^{-ak} \|e^{it(-\Delta)^{a/2}} P_0(f(2^{-k}\cdot))\|_{L^2(w(2^{-k}x, 2^{-ak}t))}^2 \\ &\leq C 2^{-kn} 2^{-ak} \|w(2^{-k}x, 2^{-ak}t)\|_{\mathfrak{L}^{\alpha,\beta,p}} \|f(2^{-k}\cdot)\|_2^2 \\ &\leq C 2^{k(\alpha+a\beta-a)} \|w\|_{\mathfrak{L}^{\alpha,\beta,p}} \|f\|_2^2. \end{aligned}$$

Now, by duality, (4.3) is equivalent to

$$\left\| \int e^{-is(-\Delta)^{a/2}} P_0 F(\cdot, s) ds \right\|_{L_x^2} \leq C \|w\|_{\mathfrak{L}^{\alpha,\beta,p}}^{1/2} \|F\|_{L^2(w^{-1})},$$

and so it is enough to show the following bilinear form estimate

$$\left| \left\langle \int_{\mathbb{R}} e^{i(t-s)(-\Delta)^{a/2}} P_0^2 F(\cdot, s) ds, G(x, t) \right\rangle \right| \leq C \|w\|_{\mathfrak{L}^{\alpha, \beta, p}} \|F\|_{L^2(w^{-1})} \|G\|_{L^2(w^{-1})}.$$

For this, let us write

$$\int_{\mathbb{R}} e^{i(t-s)(-\Delta)^{a/2}} P_0^2 F(\cdot, s) ds = K * F,$$

where

$$K(x, t) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t|\xi|^a)} \phi(|\xi|)^2 d\xi.$$

Let $\psi_j : \mathbb{R}^{n+1} \rightarrow [0, 1]$ be a smooth function which is supported in $B(0, 2^j) \setminus B(0, 2^{j-2})$ for $j \geq 1$ and in $B(0, 1)$ for $j = 0$, such that $\sum_{j \geq 0} \psi_j = 1$. Then, we decompose the kernel K into

$$K = \sum_{j \geq 0} \psi_j K,$$

and will show that

$$\sum_{j \geq 0} |\langle (\psi_j K) * F, G \rangle| \leq C \|w\|_{\mathfrak{L}^{\alpha, \beta, p}} \|F\|_{L^2(w^{-1})} \|G\|_{L^2(w^{-1})} \quad (4.4)$$

which implies the above bilinear form estimate.

To show (4.4), we will obtain the following three estimates

$$|\langle (\psi_j K) * F, G \rangle| \leq C 2^{j(\frac{n+2}{2} - (\alpha + \beta)p)} \|w\|_{\mathfrak{L}^{\alpha, \beta, p}}^p \|F\|_{L^2(w^{-p})} \|G\|_{L^2(w^{-p})}, \quad (4.5)$$

$$|\langle (\psi_j K) * F, G \rangle| \leq C 2^{j(\frac{n+2}{2} - \frac{\alpha + \beta}{2}p)} \|w\|_{\mathfrak{L}^{\alpha, \beta, p}}^{p/2} \|F\|_{L^2(w^{-p})} \|G\|_2, \quad (4.6)$$

$$|\langle (\psi_j K) * F, G \rangle| \leq C 2^{j(\frac{n+2}{2} - \frac{\alpha + \beta}{2}p)} \|w\|_{\mathfrak{L}^{\alpha, \beta, p}}^{p/2} \|F\|_2 \|G\|_{L^2(w^{-p})}. \quad (4.7)$$

Assuming these estimates for the moment, let us show the estimate (4.4) by making use of the bilinear interpolation lemma, Lemma 3.1. First, define the bilinear vector-valued operator T by

$$T(F, G) = \{\langle (\psi_j K) * F, G \rangle\}_{j \geq 0}.$$

Then (4.4) is equivalent to

$$T : L^2(w^{-1}) \times L^2(w^{-1}) \rightarrow \ell_1^0(\mathbb{C}) \quad (4.8)$$

with the operator norm $C\|w\|_{\mathfrak{L}^{\alpha, \beta, p}}$. Here, for $a \in \mathbb{R}$ and $1 \leq p \leq \infty$, $\ell_p^a(\mathbb{C})$ denotes the weighted sequence space with the norm

$$\|\{x_j\}_{j \geq 0}\|_{\ell_p^a} = \begin{cases} \left(\sum_{j \geq 0} 2^{jap} |x_j|^p \right)^{\frac{1}{p}}, & \text{if } p \neq \infty, \\ \sup_{j \geq 0} 2^{ja} |x_j|, & \text{if } p = \infty. \end{cases}$$

Next, note that the above three estimates (4.5), (4.6) and (4.7) become

$$\begin{aligned} \|T(F, G)\|_{\ell_{\infty}^{\beta_0}(\mathbb{C})} &\leq C \|w\|_{\mathfrak{L}^{\alpha, \beta, p}}^p \|F\|_{L^2(w^{-p})} \|G\|_{L^2(w^{-p})}, \\ \|T(F, G)\|_{\ell_{\infty}^{\beta_1}(\mathbb{C})} &\leq C \|w\|_{\mathfrak{L}^{\alpha, \beta, p}}^{p/2} \|F\|_{L^2(w^{-p})} \|G\|_2, \end{aligned} \quad (4.9)$$

$$\|T(F, G)\|_{\ell_{\infty}^{\beta_1}(\mathbb{C})} \leq C \|w\|_{\mathfrak{L}^{\alpha, \beta, p}}^{p/2} \|F\|_2 \|G\|_{L^2(w^{-p})}, \quad (4.10)$$

respectively, with $\beta_0 = -(\frac{n+2}{2} - (\alpha + \beta)p)$ and $\beta_1 = -(\frac{n+2}{2} - \frac{\alpha+\beta}{2}p)$. Then, applying Lemma 3.1 with $\theta_0 = \theta_1 = 1/p'$ and $q = r = 2$, one can get for $1 < p < 2$

$$T : (L^2(w^{-p}), L^2)_{1/p', 2} \times (L^2(w^{-p}), L^2)_{1/p', 2} \rightarrow (\ell_\infty^{\beta_0}(\mathbb{C}), \ell_\infty^{\beta_1}(\mathbb{C}))_{2/p', 1}$$

with the operator norm $C\|w\|_{\mathfrak{L}^{\alpha, \beta, p}}$. Finally, we use the following real interpolation space identities in Lemma 4.2 in order to obtain (4.8). Indeed, from the lemma one can see that

$$(L^2(w^{-p}), L^2)_{1/p', 2} = L^2(w^{-1})$$

for $1 < p < 2$, and

$$(\ell_\infty^{\beta_0}(\mathbb{C}), \ell_\infty^{\beta_1}(\mathbb{C}))_{2/p', 1} = \ell_1^0(\mathbb{C})$$

if $(1 - \frac{2}{p'})\beta_0 + \frac{2}{p'}\beta_1 = 0$ (i.e., $\alpha + \beta = \frac{n+2}{2}$). Hence, we get (4.8) if $\alpha + \beta = \frac{n+2}{2}$ and $1 < p < 2$.

Lemma 4.2. (cf. Theorems 5.4.1 and 5.6.1 in [3]) *Let $0 < \theta < 1$. Then one has*

$$(L^2(w_0), L^2(w_1))_{\theta, 2} = L^2(w), \quad w = w_0^{1-\theta} w_1^\theta,$$

and for $1 \leq q_0, q_1, q \leq \infty$

$$(\ell_{q_0}^{s_0}, \ell_{q_1}^{s_1})_{\theta, q} = \ell_q^s, \quad s = (1 - \theta)s_0 + \theta s_1.$$

When $\alpha + \beta > \frac{n+2}{2}$, we note that $\gamma := \frac{p'}{2}(\alpha + \beta - \frac{n+2}{2}) > 0$. Since $j \geq 0$ and $\beta_1 < 0$, the estimates (4.9) and (4.10) are trivially satisfied for β_1 replaced by $\beta_1 - \gamma$. Thus, by the same argument we only need to check that $(1 - \frac{2}{p'})\beta_0 + \frac{2}{p'}(\beta_1 - \gamma) = 0$ which is an easy computation. Consequently, we get (4.8) if $\alpha + \beta \geq \frac{n+2}{2}$ and $1 < p < 2$. This completes the proof for (4.1).

It remains only to show the three estimates (4.5), (4.6) and (4.7). For $j \geq 0$, let $\{Q_\lambda\}_{\lambda \in 2^j \mathbb{Z}^{n+1}}$ be a collection of cubes $Q_\lambda \subset \mathbb{R}^{n+1}$ centered at λ with side length 2^j . Then by disjointness of cubes, it is easy to see that

$$\begin{aligned} |\langle (\psi_j K) * F, G \rangle| &\leq \sum_{\lambda, \mu \in 2^j \mathbb{Z}^{n+1}} |\langle (\psi_j K) * (F \chi_{Q_\lambda}), G \chi_{Q_\mu} \rangle| \\ &\leq \sum_{\lambda \in 2^j \mathbb{Z}^{n+1}} |\langle (\psi_j K) * (F \chi_{Q_\lambda}), G \chi_{\tilde{Q}_\lambda} \rangle|, \end{aligned}$$

where \tilde{Q}_λ denotes the cube with side length 2^{j+2} and the same center as Q_λ . By Young's and Cauchy-Schwartz inequalities, it follows now that

$$\begin{aligned} |\langle (\psi_j K) * F, G \rangle| &\leq \sum_{\lambda \in 2^j \mathbb{Z}^{n+1}} \|(\psi_j K) * (F \chi_{Q_\lambda})\|_\infty \|G \chi_{\tilde{Q}_\lambda}\|_1 \\ &\leq \sum_{\lambda \in 2^j \mathbb{Z}^{n+1}} \|\psi_j K\|_\infty \|F \chi_{Q_\lambda}\|_1 \|G \chi_{\tilde{Q}_\lambda}\|_1 \\ &\leq \|\psi_j K\|_\infty \left(\sum_{\lambda \in 2^j \mathbb{Z}^{n+1}} \|F \chi_{Q_\lambda}\|_1^2 \right)^{\frac{1}{2}} \left(\sum_{\lambda \in 2^j \mathbb{Z}^{n+1}} \|G \chi_{\tilde{Q}_\lambda}\|_1^2 \right)^{\frac{1}{2}}. \quad (4.11) \end{aligned}$$

To estimate the first term $\|\psi_j K\|_\infty$ in the right-hand side of (4.11), we will use the following well-known lemma³, Lemma 4.3. In fact, by applying the lemma with $\varphi(\xi) = |\xi|^a$, $a > 1$, it follows that

$$|K(x, t)| = \left| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t|\xi|^a)} \phi(|\xi|)^2 d\xi \right| \leq C(1 + |(x, t)|)^{-\frac{a}{2}},$$

since the Hessian matrix $H\psi$ has n non-zero eigenvalues for each $\xi \in \{\xi \in \mathbb{R}^n : |\xi| \sim 1\}$. This implies the estimate

$$\|\psi_j K\|_\infty \leq C 2^{-j \frac{a}{2}}. \quad (4.12)$$

Lemma 4.3. *Let $H\varphi$ be the Hessian matrix given by $(\frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j})$. Suppose that η is a compactly supported smooth function on \mathbb{R}^n and φ is a smooth function satisfying $\text{rank } H\varphi \geq k$ on the support of η . Then, for $(x, t) \in \mathbb{R}^{n+1}$*

$$\left| \int e^{i(x \cdot \xi + t\varphi(\xi))} \eta(\xi) d\xi \right| \leq C(1 + |(x, t)|)^{-\frac{k}{2}}.$$

The other terms are estimated using Hölder's inequality, as follows:

$$\begin{aligned} \sum_{\lambda \in 2^j \mathbb{Z}^{n+1}} \|F\chi_{Q_\lambda}\|_1^2 &= \sum_{\lambda \in 2^j \mathbb{Z}^{n+1}} \left(\int_{Q_\lambda} |F\chi_{Q_\lambda}| w^{-\frac{p}{2}} w^{\frac{p}{2}} dx dt \right)^2 \\ &\leq \sum_{\lambda \in 2^j \mathbb{Z}^{n+1}} \left(\int_{Q_\lambda} |F\chi_{Q_\lambda}|^2 w^{-p} dx dt \right) \left(\int_{Q_\lambda} w^p dx dt \right) \\ &\leq \sup_{\lambda \in 2^j \mathbb{Z}^{n+1}} \left(\int_{Q_\lambda} w^p dx dt \right) \sum_{\lambda \in 2^j \mathbb{Z}^{n+1}} \left(\int_{Q_\lambda} |F\chi_{Q_\lambda}|^2 w^{-p} dx dt \right) \\ &\leq C 2^{j(n+1-(\alpha+\beta)p)} \|w\|_{\mathfrak{L}^{\alpha, \beta, p}}^p \|F\|_{L^2(w^{-p})}^2 \end{aligned} \quad (4.13)$$

while

$$\begin{aligned} \sum_{\lambda \in 2^j \mathbb{Z}^{n+1}} \|F\chi_{Q_\lambda}\|_1^2 &\leq \sum_{\lambda \in 2^j \mathbb{Z}^{n+1}} \|F\chi_{Q_\lambda}\|_2^2 \|\chi_{Q_\lambda}\|_2^2 \\ &\leq C 2^{j(n+1)} \|F\|_2^2. \end{aligned} \quad (4.14)$$

Similarly for $\sum_{\lambda \in 2^j \mathbb{Z}^{n+1}} \|G\chi_{\tilde{Q}_\lambda}\|_1^2$.

By combining (4.11), (4.12), (4.13) and (4.14), one can easily get the desired three estimates (4.5), (4.6) and (4.7).

Let us now turn to the second estimate (4.2) in the proposition. We will show the following estimate

$$\left\| \int_{-\infty}^t e^{i(t-s)(-\Delta)^{a/2}} F(\cdot, s) ds \right\|_{L^2(w(x, t))} \leq C \|w\|_{\mathfrak{L}^{\alpha, \beta, p}} \|F\|_{L^2(w(x, t)^{-1})} \quad (4.15)$$

which implies (4.2). In fact, to obtain (4.2) from (4.15), first decompose the L_t^2 norm in the left-hand side of (4.2) into two parts, $t \geq 0$ and $t < 0$. Then the latter can be reduced to the former by changing the variable $t \mapsto -t$, and so it is only needed to consider the first part $t \geq 0$. But, since $[0, t) = (-\infty, t) \cap [0, \infty)$, applying (4.15) with

³It is essentially due to Littman [21]. See also [29], VIII, Section 5, B.

F replaced by $\chi_{[0,\infty)}(s)F$, one can bound the first part as desired. To show (4.15), by duality we may show the following bilinear form estimate as before:

$$\left| \left\langle \int_{-\infty}^t e^{i(t-s)(-\Delta)^{a/2}} P_0^2 F(\cdot, s) ds, G(x, t) \right\rangle \right| \leq C \|w\|_{\mathcal{L}^{\alpha, \beta, p}} \|F\|_{L^2(w^{-1})} \|G\|_{L^2(w^{-1})}.$$

Let us write

$$\begin{aligned} \int_{-\infty}^t e^{i(t-s)(-\Delta)^{a/2}} P_0 F(\cdot, s) ds &= \int_{\mathbb{R}} \chi_{(0,\infty)}(t-s) e^{i(t-s)(-\Delta)^{a/2}} P_0 F(\cdot, s) ds \\ &= K * F, \end{aligned}$$

where

$$K(x, t) = \int_{\mathbb{R}^n} \chi_{(0,\infty)}(t) e^{i(x \cdot \xi + t|\xi|^a)} \phi(|\xi|) d\xi.$$

Then, it is clear that the above bilinear estimate would follow from the same argument for the homogeneous part (4.1). So we omit the details. \square

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